

**CONTRACTION OF THE  $G_{r,s}$  QUANTUM GROUP TO ITS NON  
STANDARD ANALOGUE AND CORRESPONDING  
COLOURED QUANTUM GROUPS**

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**Abstract**

The quantum group  $G_{r,s}$  provides a realisation of the two parameter quantum  $GL_{p,q}(2)$  which is known to be related to the two parameter non standard  $GL_{h,h'}(2)$  group via the contraction method. We apply the contraction procedure to  $G_{r,s}$  and obtain a new Jordanian quantum group  $G_{m,k}$ . Furthermore, we provide a realisation of  $GL_{h,h'}(2)$  in terms of  $G_{m,k}$ . The contraction procedure is then extended to the coloured quantum group  $GL_r^{\lambda,\mu}(2)$  to yield a new Jordanian quantum group  $GL_m^{\lambda,\mu}(2)$ . Both  $G_{r,s}$  and  $G_{m,k}$  are then generalised to their coloured versions which inturn provide similar realisations of  $GL_r^{\lambda,\mu}(2)$  and  $GL_m^{\lambda,\mu}(2)$ .

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# I Introduction

In recent years, a lot of interest has been generated in the study of Non Standard (or Jordanian) deformations of Lie groups and algebras. For  $GL(2)$ , the Jordanian deformation (also known as the *h-deformation*) was initially introduced in [1,2] with its two parametric generalisation given by Aghamohammadi in [3]. This was extended to the supersymmetric case  $GL(1/1)$  in [4]. At the algebra level, the non standard deformation  $U_h(sl(2))$  of  $sl(2)$  was first proposed by Ohn [5], the universal  $R$ -matrix was presented in [6-8] and irreducible representations studied in [9,10]. A peculiar feature of this deformation is that the corresponding  $R$ -matrix is triangular i.e.  $R^{-1} = R_{21}$ . The group  $GL(2)$  admits two distinct deformations with central determinant:  $GL_q(2)$  and  $GL_h(2)$ , and these are the only possible such deformations (up to isomorphism) [11]. In [12], an observation was made that the  $h$ -deformation could be obtained by a singular limit of a similarity transformation from the  $q$ -deformations of the group  $GL(2)$ . Given this contraction procedure, it would be useful to look for Jordanian deformations of other  $q$ -groups.

The two parameter quantum group  $G_{r,s}$  was proposed by Basu-Mallick in [13] as a particular quotient of the multiparameter  $q$ -deformation of  $GL(3)$ . The structure of  $G_{r,s}$  is interesting because it contains the one parameter  $q$ -deformation of  $GL(2)$  as a Hopf subalgebra and also gives a simple realisation of the quantum group  $GL_{p,q}(2)$  in terms of the generators of  $G_{r,s}$ . As an initial step in the further study of this quantum group, the authors have recently shown [15] that the dual Hopf algebra to  $G_{r,s}$  may be realised using the method described by Sudbery [14] i.e. as the algebra of tangent vectors at the identity. As well as this, a bicovariant differential calculus on  $G_{r,s}$  has also been constructed by the authors [16]. In the present paper, we investigate the contraction procedure on  $G_{r,s}$  in order to obtain its non standard counterpart. The generators of the

contracted structure are employed to realise the two parameter non standard deformation  $GL_{h,h'}(2)$ . This is similar to what happens in the  $q$ -deformed case. Furthermore, we extend the contraction procedure to the case of ‘coloured’ quantum groups and obtain a new single parameter non-standard quantum group  $GL_m^{\lambda,\mu}(2)$ .  $G_{m,k}$  is then extended to its coloured version, obtained by contraction from the coloured  $G_{r,s}$ . These new coloured extension also provide realisations of single parameter coloured quantum and Jordanian deformation of  $GL(2)$ .

In Section II, the block diagonal form of the  $R$ -matrix of  $G_{r,s}$  is presented. The contraction on this block diagonal  $R$ -matrix is carried out in Section III yielding the new  $R$ -matrix for  $G_{m,k}$ . Section IV defines a new non standard group  $G_{m,k}$  and Section V provides a new realisation of the well known  $GL_{h,h'}(2)$ . Contraction for coloured quantum groups and coloured extensions of  $G_{r,s}$  and  $G_{m,k}$  are given in Section VI with Section VII detailing the coloured realisations. In Section VIII, we summarise our results and briefly discuss possible further work. Appendices *A,B* and *C* summarise the Hopf algebra relations for  $G_{r,s}$ ,  $GL_{h,h'}(2)$  and coloured  $GL_m^{\lambda,\mu}(2)$  respectively.

## II The $G_{r,s}$ $R$ -matrix

The quantum group  $G_{r,s}$  is generated by the matrix of generators

$$T = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & f \end{pmatrix}$$

where the first four generators  $a$ ,  $b$ ,  $c$  and  $d$  form a Hopf subalgebra which is isomorphic to  $GL_q(2)$  quantum group with deformation parameter  $q = r^{-1}$ . The two parameter  $GL_{p,q}(2)$  can also be realised through the generators of this  $G_{r,s}$  Hopf algebra provided the sets of deformation parameters  $(p, q)$  and  $(r, s)$  are related to each other in a particular fashion. This quantum group can,

therefore, be used to realise both  $GL_q(2)$  and  $GL_{p,q}(2)$  quantum groups. The expression for the  $R$ -matrix of  $G_{r,s}$  is given in [13]. Explicitly, this reads

$$R(G_{r,s}) = \begin{pmatrix} r & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & r - r^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & s & 0 & 0 & 0 & r - r^{-1} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & r & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & r - r^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & s^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r \end{pmatrix}$$

with entries labelled in the usual numerical order (11), (12), (13), (21), (22), (23), (31), (32), (33). We start with the observation, first made by Aschieri and Castellani [17,18], that if we reorder the indices of this  $R$ -matrix with the elements in the order (11), (12), (21), (22), (13), (23), (31), (32), (33), then we obtain a block matrix, say  $R_q$  which is similar to the form of the  $GL_q(2)$   $R$ -matrix with the  $q$  in the  $R_{11}^{11}$  position itself replaced by the  $GL_q(2)$   $R$ -matrix.

$$R_q = \begin{pmatrix} R(GL_r(2)) & 0 & 0 & 0 \\ 0 & S & \lambda I & 0 \\ 0 & 0 & S^{-1} & 0 \\ 0 & 0 & 0 & r \end{pmatrix}$$

where  $R(GL_r(2))$  is the  $4 \times 4$   $R$ -matrix for  $GL_q(2)$  with  $q = r$ ,  $\lambda = r - r^{-1}$ ,  $I$  is the  $2 \times 2$  identity matrix and  $S$  is the  $2 \times 2$  matrix  $S = \begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix}$  where  $r$  and  $s$  are the deformation parameters. The zeroes are the zero matrices of appropriate order. The usual block structure of the  $R$ -matrix is clearly seen in this form.

It is straightforward to check that the  $RTT$ -relations with this new  $R$ -matrix give the known  $G_{r,s}$  commutation relations.

### III $R$ -matrix Contraction

It is well known [12] that the non standard  $R$ -matrix  $R_h(2)$  can be obtained from the  $q$ -deformed  $R_q(2)$  as a singular limit of a similarity transformation

$$R_h(2) = \lim_{q \rightarrow 1} (g^{-1} \otimes g^{-1}) R_q(2) (g \otimes g)$$

where  $g = \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix}$ . Such a transformation has been generalised to higher dimensions [19] and has also been successfully applied to two parameter quantum groups. Here we apply the above transformation for our  $G_{r,s}$  quantum group. Our starting point is the block diagonal form of the  $G_{r,s}$   $R$ -matrix, denoted  $R_q$

$$R_q = \begin{pmatrix} r & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & s^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r \end{pmatrix}$$

where  $\lambda = r - r^{-1}$ . We apply to  $R_q$  the transformation

$$(G^{-1} \otimes G^{-1}) R_q (G \otimes G)$$

Here the transformation matrix  $G$  is a  $3 \times 3$  matrix and chosen in the block diagonal form

$$G = \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$$

where  $g$  is the transformation matrix for the two dimensional case. The similarity transformation gives the matrix

$$\begin{pmatrix} r & \eta(r-1) & \eta(r-\lambda-1) & \eta^2(2(r-1)-\lambda) & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \lambda & \eta(\lambda+1-r) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \eta(1-r) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s & \eta(s-1) & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & s^{-1} & \eta(s^{-1}-1) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r \end{pmatrix}$$

Following the procedure outlined in [12], we substitute  $\eta = \frac{m}{1-r}$  to obtain

$$\begin{pmatrix} r & -m & mr^{-1} & m^2r^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & r-r^{-1} & -mr^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & m & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s & -m\frac{1-s}{1-r} & r-r^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & r-r^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & s^{-1} & ms^{-1}\frac{1-s}{1-r} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r \end{pmatrix}$$

In the limit  $r \rightarrow 1$ ,  $s \rightarrow 1$  such that  $\frac{1-s}{1-r} \rightarrow \frac{k}{m}$ , this yields the Jordanian  $R$ -matrix

$$R_h = R(G_{m,k}) = \begin{pmatrix} 1 & -m & m & m^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -m & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & m & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -k & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where the entries are labelled in the block diagonal form (11), (12), (21), (22), (13), (23), (31), (32), (33). It is straightforward to verify that this  $R$ -matrix is triangular and a solution of the Quantum Yang Baxter Equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

It is interesting to note that the block diagonal form of  $R(G_{m,k})$  embeds in the top left corner the  $R$ -matrix for the single parameter deformed  $GL_h(2)$  for  $m = h$ .

## IV The Non Standard $G_{m,k}$

The contracted  $R$ -matrix  $R(G_{m,k})$  can be used in conjunction with a  $T$ -matrix of generators of the form

$$T = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & f \end{pmatrix}$$

to form a two parameter non standard quantum group  $G_{m,k}$ . The  $RTT$ - relations

$$RT_1T_2 = T_2T_1R$$

(where  $T_1 = T \otimes \mathbf{1}$ ,  $T_2 = \mathbf{1} \otimes T$ ) give the commutation relations between the generators  $a, b, c, d$  and  $f$ .

$$[c, d] = -mc^2, \quad [c, b] = -m(ac + cd) = -m(ca + dc)$$

$$[c, a] = -mc^2, \quad [d, a] = -m(d - a)c = -mc(d - a)$$

$$[d, b] = -m(d^2 - \delta)$$

$$[b, a] = -m(\delta - a^2)$$

and

$$[f, a] = kcf, \quad [f, b] = k(df - fa)$$

$$[f, c] = 0, \quad [f, d] = -kcf$$

The element  $\delta = ad - bc + mac = ad - cb - mcd$  is central in the whole algebra. (Note that the first set of the above relations consist of elements  $a, b, c$  and  $d$  which form a subalgebra that coincides exactly with the single parameter non standard  $GL_h(2)$  for  $m = h$ .) The coalgebra structure of  $G_{m,k}$  can be written as

$$\Delta \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & f \end{pmatrix} = \begin{pmatrix} a \otimes a + b \otimes c & a \otimes b + b \otimes d & 0 \\ c \otimes a + d \otimes c & c \otimes b + d \otimes d & 0 \\ 0 & 0 & f \otimes f \end{pmatrix}$$

$$\varepsilon \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & f \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



Adjoining the element  $\delta^{-1}$  to the algebra enables determination of the antipode matrix  $S(T)$ ,

$$S \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & f \end{pmatrix} = \delta^{-1} \begin{pmatrix} d + mc & -b + m(d - a) + m^2c & 0 \\ -c & a - mc & 0 \\ 0 & 0 & \delta f^{-1} \end{pmatrix}$$

(The Hopf structure of  $\delta^{-1}$  is  $\Delta(\delta^{-1}) = \delta^{-1} \otimes \delta^{-1}$ ,  $\varepsilon(\delta^{-1}) = \mathbf{1}$ ,  $S(\delta^{-1}) = \delta$ .)

These relations are consistent with the usual axioms of the Hopf algebra,

$$\begin{aligned} \mu \circ (id \otimes \mu) &= \mu \circ (\mu \otimes id) \quad , \quad (id \otimes \Delta) \circ \Delta = (\Delta \otimes id) \circ \Delta \\ (id \otimes \varepsilon) \circ \Delta &= (\varepsilon \otimes id) \circ \Delta = id \quad , \quad \mu \circ (id \otimes S) \circ \Delta = \mu \circ (S \otimes id) \circ \Delta = \mathbf{1} \circ \varepsilon \\ \Delta(xy) &= \Delta(x)\Delta(y), \quad \varepsilon(xy) = \varepsilon(x)\varepsilon(y), \quad S(xy) = S(y)S(x) \end{aligned}$$

where  $\mu$  denotes the multiplication operation  $\mu(x \otimes y) = xy$  and  $id$  is the identity transformation.

It is evident that the elements  $a$ ,  $b$ ,  $c$  and  $d$  of  $G_{m,k}$  form a Hopf subalgebra which coincides with non standard  $GL(2)$  with deformation parameter  $m$ . This is exactly analogous to the  $q$ -deformed case where the first four elements of  $G_{r,s}$  form the  $GL_q(2)$  Hopf subalgebra. Again, the remaining fifth element  $f$  generates the  $GL(1)$  group, as it did in the  $q$ -deformed case, and the second parameter appears only through the cross commutation relations between  $GL_m(2)$  and  $GL(1)$  elements. Therefore,  $G_{m,k}$  can also be considered as a two parameter Jordanian deformation of classical  $GL(2) \otimes GL(1)$  group.

## V A Realisation of $GL_{h,h'}(2)$

Now we investigate the connection of the newly defined  $G_{m,k}$  with the non standard two parameter  $GL_{h,h'}(2)$ . It was observed by Basu-Mallick [13] that there is a Hopf algebra homomorphism  $\mathcal{F}$  from  $G_{r,s}$  to  $GL_{p,q}(2)$  given by

$$\mathcal{F}_N : G_{r,s} \longmapsto GL_{p,q}(2)$$

$$\mathcal{F}_N : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = f^N \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The elements  $a', b', c'$  and  $d'$  are the generators of  $GL_{p,q}(2)$  and  $N$  is a fixed non-zero integer. The relation between the deformation parameters  $(p, q)$  and  $(r, s)$  is given by

$$p = r^{-1} s^N \quad \text{and} \quad q = r^{-1} s^{-N}$$

A Hopf algebra homomorphism of exactly the same form exists between the generators of  $G_{m,k}$  and  $GL_{h,h'}(2)$  which is straightforward to verify. Moreover, the two sets of deformation parameters  $(h, h')$  and  $(m, k)$  are related via the equation

$$m = -h + Nk = -h' - Nk$$

$$\text{i.e.} \quad h = -m + Nk \quad \text{and} \quad h' = -m - Nk$$

Note that for vanishing  $k$ , have  $h' = h$  and one gets the one parameter case. In addition, using the above realisation together with the coproduct, counit and antipode axioms for the  $G_{m,k}$  algebra and the respective homeomorphism properties, one can easily recover the standard coproduct, counit and antipode for  $GL_{h,h'}(2)$ . Thus, the non standard  $GL_{h,h'}(2)$  group can in fact be reproduced from the newly defined non standard  $G_{m,k}$ . The above realisations can be exhibited in the following commutative diagram

$$\begin{array}{ccc} G_{r,s} & \xrightarrow{\mathcal{F}} & GL_{p,q}(2) \\ \text{contraction} \downarrow & & \downarrow \text{contraction} \\ G_{m,k} & \xrightarrow{\mathcal{F}} & GL_{h,h'}(2) \end{array}$$

It is curious to note that if we write  $p = e^h$ ,  $q = e^{h'}$ ,  $r = e^m$  and  $s = e^k$ , then the relations between the parameters in the  $q$ -deformed case and the  $h$ -deformed case are identical.

## VI Coloured Quantum Groups

The standard quantum group relations can be extended by parametrising the corresponding generators using some continuous ‘colour’ variables and redefining the associated algebra and coalgebra in a way that all Hopf algebraic properties remain preserved [13, 20, 21]. For the case of a single parameter quantum deformation of  $GL(2)$  (with deformation parameter  $r$ ), its ‘coloured’ version [13] is given by the  $R$ -matrix

$$R_r^{\lambda,\mu} = \begin{pmatrix} r^{1-(\lambda-\mu)} & 0 & 0 & 0 \\ 0 & r^{\lambda+\mu} & r - r^{-1} & 0 \\ 0 & 0 & r^{-(\lambda+\mu)} & 0 \\ 0 & 0 & 0 & r^{1+(\lambda-\mu)} \end{pmatrix}$$

which satisfies

$$R_{12}^{\lambda,\mu} R_{13}^{\lambda,\nu} R_{23}^{\mu,\nu} = R_{23}^{\mu,\nu} R_{13}^{\lambda,\nu} R_{12}^{\lambda,\mu}$$

the so-called ‘Coloured’ Quantum Yang Baxter Equation (CQYBE). This gives rise to the coloured  $RTT$  relations

$$R_r^{\lambda,\mu} T_{1\lambda} T_{2\mu} = T_{2\mu} T_{1\lambda} R_r^{\lambda,\mu}$$

(where  $T_{1\lambda} = T_\lambda \otimes \mathbf{1}$  and  $T_{2\mu} = \mathbf{1} \otimes T_\mu$ ) in which the entries of the  $T$  matrices carry colour dependence. The coproduct and counit for the coalgebra structure are given by

$$\begin{aligned} \Delta(T_\lambda) &= T_\lambda \otimes T_\lambda \\ \varepsilon(T_\lambda) &= \mathbf{1} \end{aligned}$$

and depend only on one colour parameter. By contrast, the algebra structure is more complicated with only generators of two different colours appearing

simultaneously in the algebraic relations. The full Hopf algebraic structure can be constructed and results in a coloured extension of the quantum group. Since  $\lambda$  and  $\mu$  are continuous variables, this implies the coloured quantum group has an infinite number of generators. The quantum determinant  $D_\lambda = a_\lambda d_\lambda - r^{-(1+2\lambda)} c_\lambda b_\lambda$  is group-like but not central, and the antipode is

$$S(T_\lambda) = D_\lambda^{-1} \begin{pmatrix} d_\lambda & -r^{1+2\lambda} b_\lambda \\ -r^{-1-2\lambda} & a_\lambda \end{pmatrix}$$

In order to investigate the contraction for coloured quantum groups. we apply to  $R_r^{\lambda,\mu}$  the transformation

$$(g \otimes g)^{-1} R_r^{\lambda,\mu} (g \otimes g)$$

where  $g$  is the two dimensional transformation matrix  $\begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix}$  and  $\eta$  is chosen to be  $\eta = \frac{m}{1-r}$ . This gives the matrix

$$\begin{pmatrix} r^{1-(\lambda-\mu)} & R_{12}^{11} & R_{21}^{11} & R_{22}^{11} \\ 0 & r^{\lambda+\mu} & r - r^{-1} & R_{22}^{12} \\ 0 & 0 & r^{-(\lambda+\mu)} & R_{22}^{21} \\ 0 & 0 & 0 & r^{1+(\lambda-\mu)} \end{pmatrix}$$

where

$$R_{12}^{11} = mr^{1-\lambda+\mu}[-1 + 2\lambda]_r$$

$$R_{21}^{11} = -mr^{1-\lambda+\mu}[1 + 2\mu]_r + mr^{-1}(1 + r)$$

$$R_{22}^{12} = mr^{\lambda+\mu}[1 - 2\mu]_r - mr^{-1}(1 + r)$$

$$R_{22}^{21} = mr^{-(\lambda+\mu)}[1 + 2\lambda]_r$$

$$R_{22}^{11} = -m^2 r([-\lambda + \mu]_r[-1 + 2\lambda]_r + [\lambda - \mu]_r[-1 - 2\lambda]_r) + m^2 r^{-1}([1 + 2\lambda]_r[1 - 2\lambda]_r)$$

and  $[x]_r = (\frac{1-x^r}{1-x})$  denotes the basic number from  $q$ -analysis.

In the limit  $r \rightarrow 1$ , we obtain a new  $R$ -matrix

$$R_m^{\lambda,\mu} = \begin{pmatrix} 1 & -m(1-2\lambda) & m(1-2\mu) & m^2(1-4\lambda\mu) \\ 0 & 1 & 0 & -m(1+2\mu) \\ 0 & 0 & 1 & m(1+2\lambda) \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which is a coloured  $R$ -matrix for a Jordanian deformation of  $GL(2)$ . This  $R$ -matrix satisfies the CQYBE and is ‘colour’ triangular i.e.  $R_{12}^{\lambda,\mu} = (R_{21}^{\mu,\lambda})^{-1}$ , a coloured extension of the notion of triangularity. This  $R$ -matrix is distinct from that of the coloured Jordanian deformation of  $GL_q(2)$  obtained in [20,21] by other means. The Hopf algebra structure and the commutation relations for the quantum group associated with this  $R$ -matrix are given in Appendix C.

**Note:** This is the first time that such a contraction procedure has been applied to obtain a coloured Jordanian  $R$ -matrix and hence the coloured Jordanian quantum group.

### VI.1 Coloured Extension of $G_{r,s} : G_r^{s,s'}$

The coloured extension of  $G_{r,s}$  proposed in [13] has only one deformation parameter  $r$  and two colour parameters  $s$  and  $s'$ . The second deformation parameter of the uncoloured case now plays the role of a colour parameter. In such a coloured extension, the first four generators  $a, b, c, d$  are kept independent of the colour parameter(s) while the fifth generator  $f$  is now parameterised by  $s$  and  $s'$ . The matrices of generators are

$$T_s = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & f_s \end{pmatrix} \quad , \quad T_{s'} = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & f_{s'} \end{pmatrix}$$

From the  $RTT$  relations, one observes that the commutation relations between  $a, b, c, d$  are as before but  $f_s$  and  $f_{s'}$  now satisfy

$$\begin{aligned} af_s &= f_s a, & bf_s &= s^{-1} f_s b, & cf_s &= s f_s c, & df_s &= f_s d \\ af_{s'} &= f_{s'} a, & bf_{s'} &= s'^{-1} f_{s'} b, & cf_{s'} &= s' f_{s'} c, & df_{s'} &= f_{s'} d \end{aligned}$$

and

$$[f_s, f_{s'}] = 0$$

Our choice of the  $9 \times 9$   $R$ -matrix for  $G_r^{s, s'}$  is

$$R_r^{s, s'} = \begin{pmatrix} R_r(2) & 0 & 0 & 0 \\ 0 & S(s, s') & 0 & 0 \\ 0 & 0 & \overline{S}(s, s') & 0 \\ 0 & 0 & 0 & r \end{pmatrix}$$

where

$$R_r(2) = \begin{pmatrix} r & 0 & 0 & 0 \\ 0 & 1 & r - r^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & r \end{pmatrix}, \quad S(s, s') = \begin{pmatrix} \sqrt{ss'} & 0 \\ 0 & \sqrt{\frac{s}{s'}} \end{pmatrix}, \quad \overline{S}(s, s') = \begin{pmatrix} \frac{1}{\sqrt{ss'}} & 0 \\ 0 & \sqrt{\frac{s}{s'}} \end{pmatrix}$$

which satisfies the CQYBE

$$R_{12}(r; s, s') R_{13}(r; s, s'') R_{23}(r; s', s'') = R_{23}(r; s', s'') R_{13}(r; s, s'') R_{12}(r; s, s')$$

$S$  and  $\overline{S}$  satisfy the exchange relation  $\overline{S}(s, s') = S(s', s)^{-1}$ .

## VI.2 Coloured Extension of $G_{m, k} : G_m^{k, k'}$

Similar to the case of  $G_{r, s}$ , we propose a coloured extension of the Jordanian quantum group  $G_{m, k}$ . The first four generators remain independent of the coloured parameters  $k$  and  $k'$  whereas the generator  $f$  is parameterised by  $k$

and  $k'$ . Again, the second deformation parameter  $k$  of the uncoloured case now plays the role of a colour parameter and the  $T$ -matrices are

$$T_k = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & f_k \end{pmatrix}, \quad T_{k'} = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & f_{k'} \end{pmatrix}$$

The commutation relations between  $a, b, c, d$  remain unchanged whereas  $f_k$  and  $f_{k'}$  satisfy

$$\begin{aligned} [f_k, a] &= kc f_k, & [f_k, b] &= k(df_k - f_k a), & [f_k, c] &= 0, & [f_k, d] &= -kc f_k \\ [f_{k'}, a] &= k'c f_{k'}, & [f_{k'}, b] &= k'(df_{k'} - f_{k'} a), & [f_{k'}, c] &= 0, & [f_{k'}, d] &= -k'c f_{k'} \end{aligned}$$

and

$$[f_k, f_{k'}] = 0$$

Our choice of the  $9 \times 9$   $R$ -matrix for  $G_m^{k, k'}$  is

$$R_m^{k, k'} = \begin{pmatrix} R_m(2) & 0 & 0 & 0 \\ 0 & K(k, k') & 0 & 0 \\ 0 & 0 & \overline{K}(k, k') & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where

$$R_m(2) = \begin{pmatrix} 1 & -m & m & m^2 \\ 0 & 1 & 0 & -m \\ 0 & 0 & 1 & m \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad K(k, k') = \begin{pmatrix} 1 & -k' \\ 0 & 1 \end{pmatrix}, \quad \overline{K}(k, k') = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$$

This  $R$ -matrix is chosen since it is the contraction limit of the  $R$ -matrix for the coloured extension of  $G_{r, s}$  via the transformation

$$R_m^{k, k'} = \lim_{r \rightarrow 1} (G \otimes G)^{-1} R_r^{s, s'} (G \otimes G)$$

where

$$G = \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}; \quad g = \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix}, \quad \eta = \frac{m}{r-1}$$

It is a solution of the CQYBE

$$R_{12}(m; k, k') R_{13}(m; k, k'') R_{23}(m; k', k'') = R_{23}(m; k', k'') R_{13}(m; k, k'') R_{12}(m; k, k')$$

and is colour triangular. Again,  $K$  and  $\overline{K}$  satisfy the exchange relation  $\overline{K}(k, k') = K(k', k)^{-1}$ .

## VII Coloured Realisations

It is shown in [13] that, similar to the uncoloured case, the coloured quantum group  $G_r^{s, s'}$  provides a realisation of the well known coloured  $GL_r^{\lambda, \mu}(2)$  where Hopf algebra homomorphism from  $G_r^{s, s'}$  to  $GL_r^{\lambda, \mu}(2)$

$$\mathcal{F}_N : G_r^{s, s'} \longmapsto GL_r^{\lambda, \mu}(2)$$

is given by

$$\mathcal{F}_N : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a'_\lambda & b'_\lambda \\ c'_\lambda & d'_\lambda \end{pmatrix} = f_s^N \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\mathcal{F}_N : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a'_\mu & b'_\mu \\ c'_\mu & d'_\mu \end{pmatrix} = f_{s'}^N \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where  $N$  is a fixed non-zero integer and the sets of colour parameters  $(s, s')$  and  $(\lambda, \mu)$  are related through quantum deformation parameter  $r$  by

$$s = r^{2N\lambda}, \quad s' = r^{2N\mu}$$

The primed generators  $a'_\lambda, b'_\lambda, c'_\lambda, d'_\lambda$  and  $a'_\mu, b'_\mu, c'_\mu, d'_\mu$  belong to  $GL_r^{\lambda, \mu}(2)$  whereas the unprimed ones  $a, b, c, d, f_s$  and  $f_{s'}$  are generators of  $G_r^{s, s'}$ .



Just as in the  $q$ -deformed case, we obtain a realisation of the  $h$ -deformed quantum group  $GL_m^{\lambda,\mu}(2)$  using the newly defined coloured quantum group  $G_m^{k,k'}$ . If we again denote the generators of  $GL_m^{\lambda,\mu}(2)$  by  $a'_\lambda, b'_\lambda, c'_\lambda, d'_\lambda$  and  $a'_\mu, b'_\mu, c'_\mu, d'_\mu$  and the generators of  $G_m^{k,k'}$  by  $a, b, c, d, f_k$  and  $f_{k'}$  then a Hopf algebra homomorphism from  $G_m^{k,k'}$  to  $GL_m^{\lambda,\mu}(2)$

$$\mathcal{F}_N : G_m^{k,k'} \mapsto GL_m^{\lambda,\mu}(2)$$

is of exactly the same form

$$\mathcal{F}_N : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a'_\lambda & b'_\lambda \\ c'_\lambda & d'_\lambda \end{pmatrix} = f_k^N \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\mathcal{F}_N : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a'_\mu & b'_\mu \\ c'_\mu & d'_\mu \end{pmatrix} = f_{k'}^N \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The sets of colour parameters  $(k, k')$  and  $(\lambda, \mu)$  are related to the Jordanian deformation parameter  $m$  by

$$Nk = -2m\lambda \quad , \quad Nk' = -2m\mu$$

and  $N$ , again, is a fixed non-zero integer.

## VIII Conclusions

In this work, we have applied the contraction procedure to the  $G_{r,s}$  quantum group and obtained a new Jordanian quantum group  $G_{m,k}$ . The group  $G_{m,k}$  has five generators and two deformation parameters and contains the single parameter  $GL_h(2)$  as a Hopf subalgebra (generated by the first four elements). The remaining fifth generator corresponds to the  $GL(1)$  group. Furthermore, we have given a realisation of the two parameter  $GL_{h,h'}(2)$  through the generators of  $G_{m,k}$  which also reproduces its full Hopf algebra structure. The results match

with the  $q$ -deformed case. The bigger picture that emerges from our analysis of contraction for uncoloured as well as coloured quantum groups and their morphisms can be represented in the following diagram

$$\begin{array}{ccccccc}
GL_{p,q}(2) & \xleftarrow{\mathcal{F}} & G_{r,s} & \xrightarrow{\mathcal{E}} & G_r^{s,s'} & \xrightarrow{\mathcal{F}} & GL_r^{\lambda,\mu}(2) \\
c \downarrow & & \downarrow c & & \downarrow c & & \downarrow c \\
GL_{h,h'}(2) & \xleftarrow{\mathcal{F}} & G_{m,k} & \xrightarrow{\mathcal{E}} & G_m^{k,k'} & \xrightarrow{\mathcal{F}} & GL_m^{\lambda,\mu}(2)
\end{array}$$

where  $\mathcal{C}$ ,  $\mathcal{F}$  and  $\mathcal{E}$  denote the contraction, Hopf algebra homomorphism and coloured extension respectively. The objects at the top level are the  $q$  deformed ones and the corresponding Jordanian counterparts are shown at the bottom level of the diagram.

Future work along the lines indicated in the paper will lead to the explicit derivation of the dual algebra for the non standard  $G_{m,k}$  and its coloured extension, and this will be presented by the authors in a later paper.

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## Appendix A - $G_{r,s}$ Quantum Group

The two parameter quantum group  $G_{r,s}$  is generated by elements  $a, b, c, d$ , and  $f$  satisfying the relations

$$\begin{aligned} ab &= r^{-1}ba, & db &= rbd \\ ac &= r^{-1}ca, & dc &= rcd \\ bc &= cb, & [a, d] &= (r^{-1} - r)bc \end{aligned}$$

and

$$\begin{aligned} af &= fa, & cf &= sfc \\ bf &= s^{-1}fb, & df &= fd \end{aligned}$$

Elements  $a, b, c, d$  satisfying the first set of commutation relations form a sub-algebra which coincides exactly with  $GL_q(2)$  when  $q = r^{-1}$ . The Hopf structure is given as

$$\Delta \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & f \end{pmatrix} = \begin{pmatrix} a \otimes a + b \otimes c & a \otimes b + b \otimes d & 0 \\ c \otimes a + d \otimes c & c \otimes b + d \otimes d & 0 \\ 0 & 0 & f \otimes f \end{pmatrix}$$

$$\varepsilon \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & f \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The Casimir operator is defined as  $\delta = ad - r^{-1}bc$ . The inverse is assumed to exist and satisfies  $\Delta(\delta^{-1}) = \delta^{-1} \otimes \delta^{-1}$ ,  $\varepsilon(\delta^{-1}) = 1$ ,  $S(\delta^{-1}) = \delta$ , which enables determination of the antipode matrix  $S(T)$ , as

$$S \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & f \end{pmatrix} = \delta^{-1} \begin{pmatrix} d & -rc & 0 \\ -r^{-1}c & a & 0 \\ 0 & 0 & \delta f^{-1} \end{pmatrix}$$

The quantum determinant  $D = \delta f$  is group-like but not central.

## Appendix B - Non Standard $GL_{h,h'}(2)$

The two parameter non standard group  $GL_{h,h'}(2)$  is generated by the matrix of generators  $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  and the  $4 \times 4$   $R$ -matrix is given as

$$R = \begin{pmatrix} 1 & -h' & h' & hh' \\ 0 & 1 & 0 & -h \\ 0 & 0 & 1 & h \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The commutation relations among the generators  $a, b, c$  and  $d$  are

$$[a, c] = hc^2, \quad [b, c] = hcd + h'ac$$

$$[d, c] = h'c^2, \quad [a, d] = hcd - h'ca$$

$$[a, b] = h'(D - a^2)$$

$$[d, b] = h(D - d^2)$$

where  $D = ad - cb - hcd = ad - bc + h'ac$  is the quantum determinant and  $h$  and  $h'$  are the two deformation parameters. The Hopf algebra structure is

$$\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \otimes a + b \otimes c & a \otimes b + b \otimes d \\ c \otimes a + d \otimes c & c \otimes b + d \otimes d \end{pmatrix}$$

$$\varepsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$D^{-1}$  exists and satisfies  $\Delta(D^{-1}) = D^{-1} \otimes D^{-1}, \varepsilon(D^{-1}) = \mathbf{1}, S(D^{-1}) = D$ .

Using this structure the antipode matrix can be expressed as

$$S(T) = D^{-1} \begin{pmatrix} d + hc & -b + h(d - a) + h^2c \\ -c & a - hc \end{pmatrix} = \begin{pmatrix} d + h'c & -b + h'(d - a) + h'^2c \\ -c & a - h'c \end{pmatrix} D^{-1}$$

## Appendix C - Coloured Jordanian $GL_m^{\lambda,\mu}(2)$

The commutation relations between the generating elements  $a_\lambda, b_\lambda, c_\lambda, d_\lambda$  and  $a_\mu, b_\mu, c_\mu, d_\mu$  of  $GL_m^{\lambda,\mu}(2)$  are

$$\begin{aligned}
[a_\lambda, c_\mu] &= m(1+2\mu)c_\lambda c_\mu \\
[d_\lambda, c_\mu] &= m(1-2\mu)c_\mu c_\lambda \\
[a_\lambda, d_\mu] &= m(1+2\mu)c_\lambda d_\mu - m(1-2\lambda)c_\mu a_\lambda \\
[a_\lambda, b_\mu] &= m(1-2\lambda)a_\lambda d_\mu - m(1-2\lambda)a_\mu a_\lambda - m(1-2\mu)c_\lambda b_\mu - m^2(1-4\lambda\mu)c_\lambda d_\mu \\
[b_\lambda, d_\mu] &= m(1+2\mu)d_\lambda d_\mu + m(1+2\lambda)c_\mu b_\lambda - m(1+2\mu)d_\mu a_\lambda + m^2(1-4\lambda\mu)c_\mu a_\lambda \\
[b_\lambda, c_\mu] &= m(1+2\mu)d_\lambda c_\mu + m(1-2\mu)c_\mu a_\lambda
\end{aligned}$$

and

$$\begin{aligned}
[a_\lambda, a_\mu] &= m(1-2\lambda)a_\lambda c_\mu - m(1-2\mu)c_\lambda a_\mu - m^2(1-4\lambda\mu)c_\lambda c_\mu \\
[b_\lambda, b_\mu] &= m(1-2\lambda)b_\lambda d_\mu + m(1+2\lambda)a_\mu b_\lambda - m(1-2\mu)d_\lambda b_\mu - m(1+2\mu)b_\mu a_\lambda \\
&\quad + m^2(1-4\lambda\mu)(a_\mu a_\lambda - d_\lambda d_\mu) \\
[c_\lambda, c_\mu] &= 0 \\
[d_\lambda, d_\mu] &= m(1+2\lambda)c_\mu d_\lambda - m(1+2\mu)d_\mu c_\lambda + m^2(1-4\lambda\mu)c_\mu c_\lambda
\end{aligned}$$

These relations satisfy the  $\lambda \leftrightarrow \mu$  exchange symmetry. The generators are arranged in the  $T$ -matrices

$$T_\lambda = \begin{pmatrix} a_\lambda & b_\lambda \\ c_\lambda & d_\lambda \end{pmatrix}, \quad T_\mu = \begin{pmatrix} a_\mu & b_\mu \\ c_\mu & d_\mu \end{pmatrix}$$

and the associated coproduct and counit are

$$\Delta(T_\lambda) = T_\lambda \otimes T_\lambda, \quad \varepsilon(T_\lambda) = \mathbf{1}$$

The quantum determinant

$$D_\lambda = a_\lambda d_\lambda - b_\lambda c_\lambda + m(1-2\lambda)a_\lambda c_\lambda = a_\lambda d_\lambda - c_\lambda b_\lambda - m(1+2\lambda)c_\lambda d_\lambda$$

satisfies  $[D_\lambda, D_\mu] \neq 0$  and  $[D_\lambda, T_\lambda] \neq 0$  unless  $\lambda = 0$ . The coalgebra structure is also invariant under the  $\lambda \leftrightarrow \mu$  exchange symmetry.